Regular N-orbits in the nilradical of a parabolic subalgebra

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Abstract. In the present paper the adjoint action of the unitriangular group in the nilradical of a parabolic subalgebra is studied. We set up general conjectures on the construction of the field of invariants and the structure of orbits of maximal dimension. The conjecture is proved for parabolic subalgebras of special types.

Consider the general linear group $\mathrm{GL}(n, K)$ defined over an algebraically closed field K of characteristic 0. Let B (N, respectively) be its Borel (maximal unipotent, respectively) subgroup, which consists of triangular matrices with nonzero (unit, respectively) elements on the diagonal. We fix a parabolic subgroup P that contains B. Denote by $\mathfrak{p}$, $\mathfrak{b}$ and $\mathfrak{n}$ the Lie subalgebras in $\mathfrak{gl}(n, K)$ that correspond to P, B and N respectively. We represent $\mathfrak{p} = \mathfrak{r} \oplus \mathfrak{m}$ as the direct sum of the nilradical $\mathfrak{m}$ and a block diagonal subalgebra $\mathfrak{r}$ with sizes of blocks $(n_1, \ldots, n_s)$. The subalgebra $\mathfrak{m}$ is invariant relative to the adjoint action of the group P, therefore, $\mathfrak{m}$ is invariant relative to the action of the subgroups B and N. We extend this action to the representation in the algebra $\mathcal{A} = K[\mathfrak{m}]$ and in the field $\mathcal{F} = K(\mathfrak{m})$. The subalgebra $\mathfrak{m}$ contains a Zariski-open P-orbit, which is called the Richardson orbit. Consequently, the algebra of invariants $\mathcal{A}^P$ coincides with K. Invariants of the adjoint action of the group N in $\mathfrak{m}$ are studied worse. In the case $P = B$, the algebra of invariants $\mathcal{A}^N$ is the polynomial algebra $K[x_{12}, x_{23}, \ldots, x_{n-1,n}]$. Let $\mathfrak{r}$ be the sum of two blocks; this case is a result of [B]. We do not know when the algebra of invariants $\mathcal{A}^N$ is finitely generated.

We formulate a number of conjectures (Conjectures 1–3) on the structure of the field of invariants $\mathcal{F}^N = \mathrm{Fract}\mathcal{A}^N$ and on the descriptions of regular N-orbits (i.e. N-orbits of maximal dimension). We prove Conjectures 1 and 2 for

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parabolic subalgebras of special types (Theorems 1–2). Theorem 3 provides a description of the field $\mathcal{F}^B$ for the same parabolic subalgebras. Propositions 1 and 2 give a partial decision of Conjecture 3. We construct the system of generators of the algebra $\mathcal{A}^N$ for a parabolic subalgebra with sizes of blocks $(2,4,2)$ (Proposition 3).

We begin with definitions. Every positive root γ in $\mathfrak{gl}(n,K)$ has the form (see [GG]) $\gamma = \varepsilon_i - \varepsilon_j$, $1 \leqslant i < j \leqslant n$. We identity a root γ with the pair (i,j) and the set of positive roots Δ^+ with the set of pairs (i,j), $i<j$. The system of positive roots $\Delta^+_{\mathfrak{r}}$ of the reductive subalgebra $\mathfrak{r}$ is a subsystem in Δ^+.

Let $\{E_{ij} : \ i<j\}$ be the standard basis in $\mathfrak{n}$. By E_γ denote the basis element E_{ij}, where $\gamma=(i,j)$.

We define a relation in Δ^+ such that $\gamma' \succ \gamma$ whenever $\gamma'-\gamma \in \Delta^+_{\mathfrak{r}}$. If $\gamma \prec \gamma'$ or $\gamma \succ \gamma'$, then the roots γ and γ' are comparable. Denote by M the set of $\gamma \in \Delta^+$ such that $E_\gamma \in \mathfrak{m}$. We identify the algebra $\mathcal{A}$ with the polynomial algebra in the variables x_{ij}, $\ (i,j)\in M$.

Definition 1. A subset S in M is called a *base* if the elements in S are not pairwise comparable and for any $\gamma \in M\setminus S$ there exists $\xi \in S$ such that $\gamma \succ \xi$.

Note that M has a unique base S, which can be constructed in the following way. We form the set S_1 of minimal elements in M (we say that γ is a minimal element in S_1 if there is no $\xi \in S_1$ such that $\gamma \succ \xi$). By definition, $S_1 \subset S$. We form a set M_1, which is obtained from M by deleting S_1 and all elements
$$\{\gamma \in M : \exists\ \xi \in S_1,\ \gamma \succ \xi\}.$$
The set of minimal elements S_2 in M_1 is also contained in S, and so on. Continuing the process, we get the base S.

Definition 2. An ordered set of positive roots $\{\gamma_1,\ldots,\gamma_s\}$ is called a *chain* if $\gamma_1=(a_1,a_2)$, $\gamma_2=(a_2,a_3)$, $\gamma_3=(a_3,a_4)$,...

Definition 3. We say that two roots $\xi,\xi' \in S$ form an *admissible pair* $q=(\xi,\xi')$ if there exists $\alpha_q \in \Delta^+_{\mathfrak{r}}$ such that the ordered set of roots $\{\xi,\alpha_q,\xi'\}$ is a chain. Note that the root α_q is uniquely determined by q.

We form the set $Q:=Q(\mathfrak{p})$ that consists of admissible pairs of roots in S. For every admissible pair $q=(\xi,\xi')$ we construct a positive root $\varphi_q=\alpha_q+\xi'$. Consider the subset $\Phi=\{\varphi_q : \ q\in Q\}$.

Let $\mathfrak{p}$ be any parabolic subalgebra. We construct a diagram by $\mathfrak{p}$, which is a square $n\times n$-matrix. Roots from S are marked by symbol $\otimes$ and roots from Φ are labeled by the symbol $\times$ in the diagram. The other entries in the diagram are empty.

Let a parabolic subalgebra $\mathfrak{p}$ be the subalgebra of type $(2,1,3,2)$ (the type of a parabolic subalgebra is the sizes of diagonal blocks). We have the following diagram.

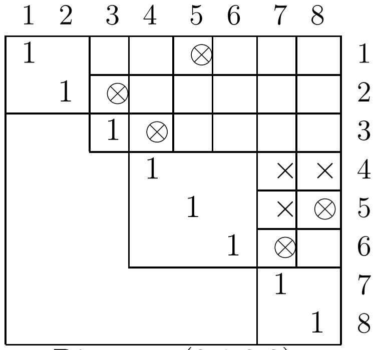

Diagram (2,1,3,2)

Consider the formal matrix $\mathbb{X}$ in which the variables x_{ij} occupy the positions $(i,j) \in M$ and the other entries are equal to zero. For any root $\gamma = (a,b) \in M$ we denote by S_γ the set of $\xi = (i,j) \in S$ such that $i > a$ and $j < b$. Let $S_\gamma = \{(i_1,j_1),\ldots,(i_k,j_k)\}$. Denote by M_γ a minor M_I^J of the matrix $\mathbb{X}$ with the ordered systems of rows $I = \mathrm{ord}\{a, i_1, \ldots, i_k\}$ and columns $J = \mathrm{ord}\{j_1,\ldots,j_k,b\}$.

For every admissible pair $q = (\xi, \xi')$, we construct the polynomial

$$L_q = \sum_{\substack{\alpha_1,\alpha_2 \in \Delta_{\mathfrak{r}}^+ \cup \{0\} \\ \alpha_1+\alpha_2=\alpha_q}} M_{\xi+\alpha_1} M_{\alpha_2+\xi'}. \tag{1}$$

Conjecture 1. The field of invariants $\mathcal{F}^N$ is the field of rational functions of polynomials M_ξ, $\xi \in S$, and L_q, $q \in Q$.

A next conjecture is a consequence of the preceding one.

Conjecture 2. The maximal dimension of an N-orbit in $\mathfrak{m}$ is equal to $\dim \mathfrak{m} - |S| - |Q|$.

Denote by $\mathcal{Y} := \mathcal{Y}_{\mathfrak{p}}$ the subset in $\mathfrak{m}$ that consists of matrices

$$\sum_{\xi \in S} c_\xi E_\xi + \sum_{\varphi \in \Phi} c'_\varphi E_\varphi.$$

Conjecture 3. Any regular N-orbit (i.e., an orbit of maximal dimension) has a nonzero intersection with $\mathcal{Y}$.

Notation. We can replace the set Φ by any similar subset Ψ in the following way. We can replace the root $\alpha_q + \xi'$ in Φ by one of two roots $\xi + \alpha_q$ and $\alpha_q + \xi'$.

Theorem 1. For an arbitrary parabolic subalgebra, the system of polynomials

$$\{M_\xi,\ \xi \in S,\ L_q,\ q \in Q, \}$$

is contained in $\mathcal{A}^N$ and is algebraically independent over K.

PROOF. The representation of P in $\mathcal{A} = K[\mathfrak{m}]$ is determined by $T_g f(x) = f(\mathrm{Ad}_g^{-1} x)$, where $g \in P$ and $f \in \mathcal{A}$. The action of T_g in $\mathcal{A}$ is uniquely defined by the action on $x_{i,j}$, $(i,j) \in M$. The elements $x_{i,j}$ make the matrix $T_g \mathbb{X} = g^{-1}\mathbb{X}g$, where the formal matrix $\mathbb{X}$ is defined above.

A polynomial f of $\mathcal{A}$ is an N-invariant if f is an invariant of the adjoint action of any one-parameter subgroup $g_k(t) = 1 + tE_{k,k+1}$, $1 \leqslant k < n$. The action of $g_i(t)$ on the matrix $\mathbb{X}$ reduces to the composition of two transformations:

1) the row with number $k+1$ multiplied by $-t$ is added to the row with number k,

2) the column with number k multiplied by t is added to the column with number $k+1$.

The invariance of $M_{(a,b)}$ follows from the notations:

1). Numbers of rows and columns of the minor $M_{(a,b)}$ fill the segments of natural numbers $I = [a, \max I]$, $J = [\min J, b]$.

2). All elements (i,j) of the matrix $\mathbb{X}$ are equal to zero, where $a \leqslant i \leqslant n$ and $1 \leqslant j < \min J$ or $\max I < i \leqslant n$ and $1 \leqslant j \leqslant b$.

Now let us prove that L_q is in $\mathcal{A}^N$. This statement follows from the invariance of L_q under the adjoint action of the one-parameter subgroup $g_k(t)$, where $g_k(t)$ corresponds to the simple root $\beta = (k, k+1)$, $1 \leqslant k < n$.

Let $q = (\xi, \xi')$, where $\xi = (a,b)$, $\xi' = (a', b')$. Using the definition of admissible pair, we have $a < b < a' < b'$ and $\alpha_q = (b, a') \in \Delta^+_{\mathfrak{r}}$. If $k < b$ or $k \geqslant a'$, then the minors of the right part of (1) are $g_k(t)$-invariants.

If $b \leqslant k < k+1 \leqslant a'$, then $\alpha_q = \gamma_1 + \beta + \gamma_2$, where $\gamma_1, \gamma_2 \in \Delta^+_{\mathfrak{r}} \cup \{0\}$. We have

$$\left\{ \begin{array}{l} T_{g_k(t)} M_{\xi+\gamma_1+\beta} = M_{\xi+\gamma_1+\beta} + tM_{\xi+\gamma_1}, \\ T_{g_k(t)} M_{\beta+\gamma_2+\xi'} = T_{\beta+\gamma_2+\xi'} - tM_{\gamma_2+\xi'}. \end{array} \right. \tag{2}$$

The other minors of (1) are invariants under the action of $g_k(t)$. Combining (1) and (2), we get

$$\left(T_{g_k(t)} L_q\right) - L_q = M_{\xi+\gamma_1}\left(M_{\beta+\gamma_2+\xi'} - tM_{\gamma_2+\xi'}\right) +$$
$$\left(M_{\xi+\gamma_1+\beta} + tM_{\xi+\gamma_1}\right) M_{\gamma_2+\xi'} - M_{\xi+\gamma_1} M_{\beta+\gamma_2+\xi'} - M_{\xi+\gamma_1+\beta} M_{\gamma_2+\xi'} = 0.$$

To prove the second statement of the theorem, we need an order relation on the set of roots $S \cup \Phi$ such that

1) $\xi < \varphi$ for any $\xi \in S$ and $\varphi \in \Phi$;

2) for other pairs of roots from $S \cup \Phi$, the relation $<$ means the lexicographic order relation.

Consider the restriction homomorphism $\pi : f \mapsto f|_{\mathcal{Y}}$ of the algebra $\mathcal{A}$ to $\mathcal{Y}$. The image of $\mathcal{A}$ is the polynomial algebra $K[\mathcal{Y}]$ of x_ξ, $\xi \in S$, and of x_φ, $\varphi \in \Phi$. The image of M_ξ, $\xi \in S$, has the form

$$\pi\left(M_\xi\right) = \pm x_\xi \prod_{\xi' \in S_\xi} x_{\xi'}. \tag{3}$$

Suppose the root $\varphi \in \Phi$ corresponds to the admissible pair $q = (\alpha, \beta)$. Then the image of the polynomial L_q has the form

$$\pi\left(L_q\right) = \pm x_\varphi x_\alpha \prod_{\alpha' \in S_\alpha} x_{\alpha'} \prod_{\beta' \in S_\alpha} x_{\alpha'}. \tag{4}$$

The system of the images $\{\pi\left(M_\xi\right), \xi \in S, \pi\left(L_q\right), q \in Q\}$ is algebraically independent over K. Therefore, the system $\{M_\xi, \xi \in S, L_q, q \in Q\}$ is algebraically independent over K. $\square$

Consider the open subset $\mathcal{U}_0 = \{x \in \mathfrak{m} : \ M_\xi \neq 0, \ \forall \xi \in S\} \subset \mathfrak{m}$.

Proposition 1. Let $\mathfrak{p}$ be a parabolic subalgebra of type $(n_1, \ldots, n_s)$. Suppose $n_1 \geqslant \ldots \geqslant n_s$; then the N-orbit of any $x \in \mathcal{U}_0$ intersects $\mathcal{Y}$ at a unique point.

PROOF. Let $n_1 \geqslant \ldots \geqslant n_s$, then the number of elements of S is equal to $n_2 + \ldots + n_s$. The set S consists of the roots $\xi_{i,j} = (m_i - j + 1, m_i + j)$, where $m_i = n_1 + \ldots + n_i$ and $1 \leqslant i \leqslant s-1$, $1 \leqslant j \leqslant n_{i+1}$.

1. Let us show that for any $A = (a_{ij}) \in \mathcal{U}_0$ there exists $g \in N$ such that $\mathrm{Ad}_g A \in \mathcal{Y}$. The proof is by induction on the number of diagonal blocks. Suppose that the statement is true for a parabolic subalgebra $\mathfrak{p}_1$, where $\mathfrak{p}_1 \subset \mathfrak{gl}(n-n_1, K)$ has the reductive subalgebra of type $(n_2 \geqslant \ldots \geqslant n_s)$. Let us show that the statement is true for a parabolic subalgebra $\mathfrak{p}$ of type $(n_1 \geqslant n_2 \geqslant \ldots \geqslant n_s)$.

   Consider the Lie algebra $\mathfrak{gl}(n - n_1, K)$ regarded as a subalgebra of $\mathfrak{gl}(n, K)$. Let $\mathfrak{gl}(n - n_1, K)$ have zeros in the first n_1 rows and columns. The systems $S_1 \subset S$ and $\Phi_1 \subset \Phi$ correspond to the parabolic subalgebra $\mathfrak{p}_1$. For the algebra $\mathfrak{p}$ of type $(2, 2, 2, 1, 1)$ and the subalgebra $\mathfrak{p}_1$ of type $(2, 2, 1, 1)$, we have the following diagrams.

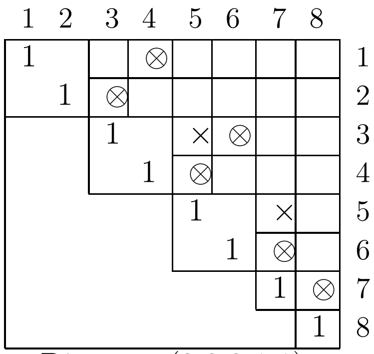

Diagram (2,2,2,1,1)

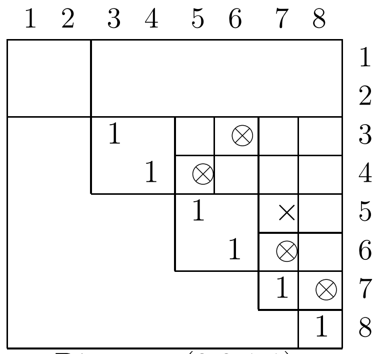

Diagram (2,2,1,1)

By the inductive assumption, all elements $a_{i,j}$ of a matrix A from the nilradical of $\mathfrak{p}$ are equal to zero for all $n_1 < i, j \leqslant n$ and $(i,j) \notin S_1 \cup \Phi_1$.

Consider the matrix $\mathrm{Ad}_{g_1} A$, where

$$g_1 = \exp\left(1 + t_1 E_{n_1+1,n_1+2} + \ldots + t_{n_2-1} E_{n_1+1,n_1+n_2} + \right.$$

$$\left. + t'_1 E_{n_1-1,n_1} + \ldots + t'_{n_1-1} E_{1,n_1}\right).$$

By the condition, $a_{n_1,n_1+1} \neq 0$. There exist $t_1, \ldots, t'_1, \ldots$ such that the elements of the matrix $\mathrm{Ad}_{g_1} A$ at the entries $(1, n_1+1), \ldots, (n_1-1, n_1+1)$ and $(n_1, n_1+2), \ldots (n_1, n_1+n_2)$ are equal to zero.

In general, elements of the matrix $\mathrm{Ad}_{g_1} A$ at the entries (i, n_1+n_2+1) are not equal to zero, where $n_1 + 1 \leqslant i \leqslant n_1 + n_2 - 1$. The entries (i, n_1+n_2+1) fill by the symbol $\times$ (see the entry (3,5) in the Diagram $(2,2,2,1,1)$).

Similarly, we find $g_2, \ldots, g_{n_2}$ such that all elements of the matrix

$$A' = \mathrm{Ad}_{g_{n_2}} \ldots \mathrm{Ad}_{g_2} \mathrm{Ad}_{g_1} A$$

are equal to zero, except for the elements of $S \cup \Phi$ or for the block

$$\{(i,j) : \ 1 \leqslant i \leqslant n_1, \ n_1 + n_2 + 1 \leqslant j \leqslant n\}. \tag{5}$$

The elements from the block (5) are labeled by the symbol $*$ in the following diagram.

$$
\begin{array}{cccccccc|c}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \\
\hline
1 & & & \otimes & * & * & * & * & 1 \\
 & 1 & \otimes & & * & * & * & * & 2 \\
 & & 1 & & \times & \otimes & & & 3 \\
 & & & 1 & \otimes & & & & 4 \\
 & & & & 1 & & \times & & 5 \\
 & & & & & 1 & \otimes & & 6 \\
 & & & & & & 1 & \otimes & 7 \\
 & & & & & & & 1 & 8
\end{array}
$$

Diagram (2,2,2,1,1)

Note that the marked by the symbol $\otimes$ elements of the matrix A' are not equal to zero.

We shall show that there is the element $h \in N$ such that the marked by $*$ elements of the matrix $\mathrm{Ad}_h A'$ are equal to zero. We start with the nth column. Let the symbol $\otimes$ be at the entry (i, n) of the last column for some i. There exist $s_1, \ldots, s_{n_1}$ such that the entries $(1, n), \ldots, (n_1, n)$ of the matrix $\mathrm{Ad}_{h_1} A'$ are equal to zero, where

$$h_1 = \exp\left(1 + s_1 E_{1,i} + \ldots + s_{n_1} E_{n_1,i}\right).$$

In the same way, we find $h_2, \ldots, h_{n-n_1}$ such that

$$\mathrm{Ad}_{h_{n-n_1}} \ldots \mathrm{Ad}_{h_2} A' \in \mathcal{Y}.$$

2. Taking into account (3) and (4), we have that the N-orbit of A intersects $\mathcal{Y}$ at a unique point. □

Proposition 2. Let $\mathfrak{p}$ be a parabolic subalgebra of type (n_1, n_2, n_3), where n_1, n_2, n_3 are any numbers. Then the N-orbit of any $x \in \mathcal{U}_0$ intersects $\mathcal{Y}$ at a unique point.

Proof. The proof is similarly. □

Let $\mathcal{S}$ be the set of denominators generated by the minors M_ξ, $\xi \in S$. We form the localization $\mathcal{A}^N_{\mathcal{S}}$ of the algebra of invariants $\mathcal{A}^N$ on $\mathcal{S}$. Since the minors M_ξ are N-invariants, we have $\mathcal{A}^N_{\mathcal{S}} = (\mathcal{A}_S)^N$.

Theorem 2. Under the conditions of Proposition 1 or 2, we have the following statements.

1. The ring $\mathcal{A}^N_S$ is the ring of polynomials in $M_\xi^{\pm 1}$, $\xi \in S$, and in L_q, $q \in Q$.

2. The field of invariants $\mathcal{F}^N$ is the field of the rational functions of M_ξ, $\xi \in S$, and L_q, $q \in Q$.

Proof. Consider the restriction homomorphism $\pi : f \mapsto f|_{\mathcal{Y}}$ of the algebra $\mathcal{A}^N$ to $K[\mathcal{Y}]$. Under the conditions of Proposition 1 or 2, the image $\pi(M_\xi)$ is equal to the product

$$\pm x_\xi x_{\xi_1} \ldots x_{\xi_s},$$

where every ξ_{i+1} is the greatest root in S, in the sense of the above order, lesser than ξ_i. We extend π to a homomorphism

$$\pi_S : \mathcal{A}_{\mathcal{S}}^N \to K[\mathcal{Y}]_S,$$

where $K[\mathcal{Y}]_S$ is the localization of $K[\mathcal{Y}]$ with respect to x_ξ, $\xi \in S$. We show that π_S is an isomorphism.

If $f \in \operatorname{Ker} \pi_S$, then $f(\mathrm{Ad}_N \mathcal{Y}) = 0$. Since, by Proposition 1 and 2, $\mathrm{Ad}_N \mathcal{Y}$ contains a Zariski-open subset, then $f = 0$. Consequently, π_S is an embedding $\mathcal{A}_{\mathcal{S}}^N$ in $K[\mathcal{Y}]_S$. Next, we write the formulates (3) and (4) in the form

$$\begin{aligned} &\pi\left(M_\xi\right) = \pm x_\xi \pi\left(M_{\xi_1}\right), \\ &\pi\left(L_q\right) = \pm x_\varphi \pi\left(M_\xi\right) \pi\left(M_{\xi_1'}\right), \end{aligned} \tag{6}$$

where the admissible pair $q = (\xi, \xi')$ corresponds to the root $\varphi \in \Phi$ and ξ_1 (ξ_1', respectively) is the greatest root in the base that is less than ξ (ξ', respectively) in the sense of the lexicographical order. We get

$$\begin{aligned} &\pi_S\left(M_\xi M_{\xi_1}^{-1}\right) = \pm x_\xi, \\ &\pi_S\left(L_q M_\xi^{-1} M_{\xi_1'}^{-1}\right) = \pm x_\varphi. \end{aligned} \tag{7}$$

From (7) it follows that the image of π_S coincides with $K[\mathcal{Y}]_S$. Thus, π_S is an isomorphism. □

By $\mathfrak{A}$ denote the system of weights α_q, $q \in Q$.

Theorem 3. Under the conditions of Proposition 1 or 2, we have the field $\mathcal{F}^B$ is the field of the rational functions of $\operatorname{corank}(\mathfrak{A})$ variables.

Proof. Consider the Cartan subgroup $H \subset \mathrm{GL}(n, K)$ of the diagonal matrices. The field of invariants $\mathcal{F}^B$ is a subfield of $\mathcal{F}^N$ and coincides with $\left(\mathcal{F}^N\right)^H$. The system of roots $S \cup \Phi$ generates the lattice of weights of the field $\mathcal{F}^N$. The field $\mathcal{F}^B$ is a transcendental extension of K and

$$\operatorname{tr}\deg \mathcal{F}^B = \operatorname{corank}(S \cup \Phi).$$

The system S is linearly independent and

$$\operatorname{rank}(S \cup \Phi) = \operatorname{rank}(S \cup \mathfrak{A}) = \operatorname{rank}(S) + \operatorname{rank}(\mathfrak{A}).$$

Further, $\operatorname{corank}(S \cup \Phi) = \operatorname{corank}(\mathfrak{A})$. $\square$

Consider the parabolic subalgebra $\mathfrak{p} \subset \mathfrak{gl}(8, K)$ with sizes of blocks $(2, 4, 2)$. We give the complete description of the algebra of invariants $\mathcal{A}^N$. The base S consists of the roots $\alpha_1 = (2, 3)$, $\alpha_2 = (1, 4)$, $\beta_1 = (6, 7)$, $\beta_2 = (5, 8)$. Any pair (α_i, β_j) is an admissible one. The parabolic subalgebra $\mathfrak{p}$ corresponds to the following diagram.

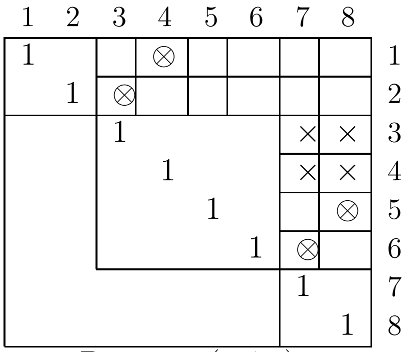

Diagram (2,4,2)

Let M_1, M_2, N_1, N_2 be the minors M_{α_1}, M_{α_2}, M_{β_1}, M_{β_2}, respectively. By $L_{i,j}$ denote the corresponding to the admissible pair (α_i, β_j) polynomial. By Theorem 2, the field of invariants $\mathcal{F}^N$ is the field of the the rational functions of M_1, M_2, N_1, N_2, $L_{1,1}$, $L_{1,2}$, $L_{2,1}$, $L_{2,2}$. The generates have the form

$$M_1 = x_{23}, \quad M_2 = \begin{vmatrix} x_{13} & x_{14} \\ x_{23} & x_{24} \end{vmatrix}, \quad N_1 = x_{67}, \quad N_2 = \begin{vmatrix} x_{57} & x_{58} \\ x_{67} & x_{68} \end{vmatrix},$$

$$L_{1,1} = x_{23}x_{37} + x_{24}x_{47} + x_{25}x_{57} + x_{26}x_{67},$$

$$L_{1,2} = \begin{vmatrix} x_{13} & x_{14} \\ x_{23} & x_{24} \end{vmatrix} x_{47} + \begin{vmatrix} x_{13} & x_{15} \\ x_{23} & x_{25} \end{vmatrix} x_{57} + \begin{vmatrix} x_{13} & x_{16} \\ x_{23} & x_{26} \end{vmatrix} x_{67},$$

$$L_{2,1} = x_{23} \begin{vmatrix} x_{37} & x_{38} \\ x_{67} & x_{68} \end{vmatrix} + x_{24} \begin{vmatrix} x_{47} & x_{48} \\ x_{67} & x_{68} \end{vmatrix} + x_{25} \begin{vmatrix} x_{57} & x_{58} \\ x_{67} & x_{68} \end{vmatrix},$$

$$L_{2,2} = \begin{vmatrix} x_{13} & x_{14} \\ x_{23} & x_{24} \end{vmatrix} \cdot \begin{vmatrix} x_{47} & x_{48} \\ x_{67} & x_{68} \end{vmatrix} + \begin{vmatrix} x_{13} & x_{15} \\ x_{23} & x_{25} \end{vmatrix} \cdot \begin{vmatrix} x_{57} & x_{58} \\ x_{67} & x_{68} \end{vmatrix}.$$

By D denote the minor $M_{1,2}^{7,8}$ of the matrix $\mathbb{X}^2$. It is easily shown that D is an N-invariant. We have the identity

$$L_{1,2}L_{2,1} - L_{1,1}L_{2,2} = M_1 N_1 D. \tag{8}$$

By $\mathcal{B}_0$ denote the subalgebra such that the polynomials M_i, N_i, $L_{i,j}$, $i,j=1,2$, generate $\mathcal{B}_0$. By $\mathcal{B}_1$ denote the subalgebra such that $\mathcal{B}_1$ is generated by $\mathcal{B}_0$ and D. Since all generators are N-invariants, we have $\mathcal{B}_0 \subset \mathcal{B}_1 \subset \mathcal{A}^N$.

Proposition 3. We have

1) $\mathcal{B}_0 \neq \mathcal{B}_1$;

2) $\mathcal{A}^N = \mathcal{B}_1$.

PROOF.

1. Suppose $\mathcal{B}_0 = \mathcal{B}_1$, then the invariant D is contained in $\mathcal{B}_0$. Therefore, there exists the polynomial $f(u_1,\ldots,u_8)$ such that

$$D = f(M_1, M_2, N_1, N_2, L_{1,1}, L_{1,2}, L_{2,1}, L_{2,2}). \tag{9}$$

Combining (8) and (9), we obtain that the system M_i, N_i, L_{ij}, $i,j \in \{1,2\}$, is algebraically dependent. This contradicts Theorem 1.

2. Let $\mathcal{S}$ be the set of denominators generated by minors M_1, M_2, N_1 and N_2. By Theorem 2, it follows that the localization $\mathcal{A}^N_{\mathcal{S}}$ of the N-algebra of invariants on $\mathcal{S}$ coincides with the algebra of Laurent polynomials

$$K[M_1^{\pm1}, M_2^{\pm1}, N_1^{\pm1}, N_2^{\pm1}, L_{11}, L_{12}, L_{21}, L_{22}].$$

If $f \in \mathcal{A}^N$, then there exist k_1, k_2, k_3, k_4 such that

$$M_1^{k_1} M_2^{k_2} N_1^{k_3} N_2^{k_4} f \in \mathcal{B}_0.$$

Let us show that for any $M \in \{M_1, M_2, N_1, N_2\}$ we have that if $F \in \mathcal{A}$ and $MF \in \mathcal{B}_1$, then $F \in \mathcal{B}_1$. From this it follows that $f \in \mathcal{B}_1$.

We proof the theorem when $M = M_1$. The cases $M = M_2$, $M = N_1$ and $M = N_2$ are similar. Let $F \in \mathcal{A}^N$ and $M_1 F \in \mathcal{B}_1$. Denote $M_1 F = h$. We have $h|_{\mathrm{Ann} M_1} = 0$. We form the matrix

$$Y := Y_{a,b,c} := \begin{pmatrix} 0 & 0 & a_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c_{11} & c_{12} \\ 0 & 0 & 0 & 0 & 0 & 0 & c_{21} & c_{22} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & b_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where a_i, b_i, c_{ij} are any numbers, $i, j \in \{1, 2\}$.

We have

$$M_1(Y) = 0, \quad M_2(Y) = -a_1a_2, \quad N_1(Y) = b_1, \quad N_2(Y) = b_1b_2,$$

$$L_{11}(Y) = a_2c_{21},\ L_{12}(Y) = -a_2b_1c_{22},$$

$$L_{21}(Y) = -a_1a_2c_{21},\ L_{22}(Y) = a_1a_2b_1c_{22},$$

$$D(Y) = a_1a_2 \begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix} = a_1a_2(c_{11}c_{22} - c_{12}c_{21}).$$

By V denote the space K^9 and by $K[V]$ denote the polynomial algebra $K[u_1, u_2, v_1, v_2, w_{11}, w_{12}, w_{21}, w_{22}, z]$. Since $h \in \mathcal{B}_1$, there exists the polynomial $p(u_1, u_2, v_1, v_2, w_{11}, w_{12}, w_{21}, w_{22}, z) \in K[V]$ such that

$$h = p(M_1, M_2, N_1, N_2, L_{11}, L_{12}, L_{21}, L_{22}, D). \tag{10}$$

Since h is equal to zero in $\operatorname{Ann} M_1$, then $h(Y) = 0$. We compute (10) at the point Y. We have

$$p(0, -a_1a_2, b_1, -b_1b_2, a_2c_{21}, -a_2b_1c_{22}, -a_1a_2c_{21}, a_1a_2b_1c_{22},$$

$$a_1a_2(c_{11}c_{22} - c_{12}c_{21})) = 0$$

for any $a_i, b_j, c_{ij} \in K$. There exist $p_1, p_2 \in K[V]$ such that

$$p = u_1p_1 + (w_{12}w_{21} - w_{11}w_{22})p_2.$$

Combining the last equation, (8) and (10), we get

$$M_1F = M_1p_1 + (L_{1,2}L_{2,1} - L_{1,2}L_{2,2})p_2 = M_1p_1 + M_1N_1Dp_2.$$

Hence, $F = p_1 + N_1Dp_2 \in \mathcal{B}_1$. $\square$

References

[R] R. W. Richardson, Conjugacy classes in parabolic subgroups of semisimple algebraic groups, *Bull. London Math. Soc.* **6** (1974), 21–24.

[B] M. Brion, Representations exceptionnelles des groups semi-simple, *Ann. Scient. Ec. Norm. Sup.* **18** (1985), 345–387.

[GG] M. Goto and F. Grosshans, Semisimple Lie algebras, Lect. Notes in Pure Appl. Math., vol. 38, 1978.

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